Latin Square Design(LSD)

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Abstract- **The Latin Square Design is one of the most important designs used in many experimentation. It provides more opportunity than Complete Randomized Design and Randomized Complete Block Design for the reduction of experimental error by skillful planning. In our seminar, we talked about some of experimental designs including Complete Randomized Design (CRD), Randomized Complete Block Design (RCBD) and Latin Square Design (LSD). We design an experiment to study the effect of different "Calculus II" lecture time on the student's marks in the first exam. We choose the marks from different classes of "Calculus II", we analyze the marks of students using SPSS program. In our work, we dwelled into the computation aspect and focused on interpretation of the results since it is much more important than the drudgery of complex computations. We used the statistical program (SPSS) for analyzing the results of our experiment. We conclude that the lecture time does not affect on the marks of students when we apply the CRD. When we use RCBD, the results were recorded and we conclude that the lecture time affect on their marks in the first exam of "Calculus II" .**

I. INTRODUCTION

Before an experiment is performed, we must address a design. In order to understand well the structure of this design and it is application, we have to discuss firstly the basic concepts of any experimental design as we will discuss in chapter one. In our seminar, we will discuss some of the common experimental designs which are the CRD and RCBD in chapter two. We will describe each of these two designs in details, the structure of these designs, the model, analysis of variance and the assumptions embodied in the model. In chapter three, we will take the case when we have two nuisance factors that we wish to control them by blocking them off in two directions in order to study the effect of a treatment variable. The design for accomplishing this is called a Latin Square Design. We introduce the description of this design and analysis of variance of the design.

II. DESIGN OF EXPERIMENTS

Much of our knowledge about products and processes in the engineering and scientific Disciplines are derived from experimentation. An experiment is a series of tests conducted in a systematic manner to increase the understanding of an existing process or to explore a new product or process. Design of Experiments, is a tool to develop an experimentation strategy that maximizes learning using a minimum of resources. Design of Experiments is widely used in many fields with broad application across all the natural and social sciences. It is extensively used by engineers and scientists involved in the improvement of manufacturing processes to maximize yield and decrease variability. Often times, engineers also work on products or processes where no scientific theory or principles are directly applicable. Experimental design techniques become extremely important in such situations to develop new products and processes in a cost-effective and confident manner.

If an experiment has been properly designed or planed, the data will have been collected in the most efficient manner for the problem being considered, and as a result the analysis of an experiment will lead to valid statistical inferences. The purpose of this chapter is to –firstly-define the experimental design concept, and other related concepts in the terminology of the design of experiment. Then, we shall introduce some basic principles of experimental design.

In order to realize the meaning of the experimental design, we need to show firstly the definition of an experiment. An experiment is a process or study that results in the collection of data. The results of experiments are not known in advance. Usually, statistical experiments are conducted in situations in which researchers can manipulate the conditions of the experiment and can control the factors that are irrelevant to the research objectives.

There are some statistical concepts that must be clarified in this context:

Experimental unit: a unit could be a person, an animal, a plant or a thing which is actually studied by a researcher; the basic objects upon which the study or experiment is carried out.

Factor: a factor of an experiment is a controlled independent variable; a variable whose levels are set by the experimenter. And a factor is a general type or category of treatments and different treatments constitute different levels of a factor.

Treatment: In an experiment, the factor (also called an independent variable) is an explanatory variable manipulated by the experimenter. Each factor has two or more levels, i.e: different values of the factor. Combinations of factor levels are called treatments.

For example, three different groups of runners are subjected to different training methods. The runners are the experimental units, the training methods, the treatments, where the three types of training methods constitute three levels of the factor 'type of training'.

In our work, we shall consider the major elements of any experimental design which are: The set of treatments included in the study. second, the set of experimental units included in the study. Third the rules and procedures by which the treatments are assigned to the experimental units (or vice versa). Finally, the measurements that are made on the experimental units after the treatment have been applied.

There are three basic principles of experimental design which ensure the validity of the analysis of an experiment and increase its sensitivity. These principles arerandomization, replication, blocking.

Randomization: the first principle of a designed experiment is randomization, which is the process by which experimental units (the basic objects upon which the study or experiment is carried out) are allocated to treatments, random process and not by any subjective and hence possibly biased approach. The treatments should be allocated to units in such a way that each treatment is equally likely to be applied to each unit.

Randomization provides protection against uncontrollable factors and to reduce or eliminate selection bias, and the randomization also can provide the basis for making inferences without requiring assumptions about the distribution of the error terms.

Replication: the second principle of a designed experiment is replication, which is a complete repetition of the basic experiments. It refers to running all the treatments combination again, where each treatment is applied to several experimental units. Consider an experiment consisting of three treatments. the assignments of three experimental units at random, one to each treatment, constitute one replication of the experiment. The assignment of an additional three experimental units at random to the three treatments constitute a second replication, and so on. Replication provides an estimate of the magnitude of the experimental error or makes it possible to assess the error mean square, and also makes tests of significance of effects possible.

Blocking: the third principle of a designed experiment is blocking, this is the procedure by which experimental units are grouped into homogeneous clusters in an attempt to improve the comparison of treatments by randomly allocating the treatments within each cluster (block).

There are many used experimental designs. Each design can be analyzed by using a specific ANOVAthat is designed for that experimental design. Most experimental designs require experimental units to be allocated to treatments either randomly or randomly with constraints, as in blocked designs. We will discuss in details some of these designs in the next two chapters after introducing the concept of blocking and randomization.

III. SIMPLE EXPERIMENTAL DESIGNS

In the preceding chapter, we introduce the concept of experimental design statistically, some basic principles in designing an experiment which lead to more sensitive analysis, and the major elements of any experimental design.

In this chapter, we shall discuss some commonly used designs for general applications. We will discuss the Completely Randomized Design (CRD) and its description. Also in which situation this design is appropriate. In such section of this chapter we shall also discuss the design and analysis of Randomized Complete Block Design (RCBD). We have to introduce when and how to conduct a Randomized Complete Block Design.

Completely Randomized Design (CRD): In the completely randomized design, the treatments are allocated entirely by chance. In other words, all experimental units are considered the same and no division or grouping among them exists. In a CRD, the treatments are allocated to the experimental units completely at random, and the randomization is performed using a random number table, computer, program, etc. The CRD should be used when the experimental units are homogeneous or missing values are expected to occur, and this design is also appropriate for experiments with a small number of treatments.

Statistical Model and Analysis for CRD: If we take n_i replications for each treatment or treatment combination in completely random manner, then the analysis of variance model for the experiment is given by

$$y_{ij} = \mu + \tau i + \varepsilon ij \ (i=1,...,a \ ; \ j=1,...,n_i)$$

where:

y_{ij} is the j-th observation of the i-th treatment.
μ is the population mean where $-\infty < \mu < \infty$.
τ_i is the effect due to i^{th} treatment.
ε_{ij} is the usual normal independent $(0,\sigma_\varepsilon{}^2)$ random error term or the error associated with the i^{th} treatment and the j^{th} observation.

Then from the model, we have $\sigma_Y{}^2 = \sigma_\tau{}^2 + \sigma_\varepsilon{}^2$ and so $\sigma_\tau{}^2$ and $\sigma_\varepsilon{}^2$ are components of $\sigma_Y{}^2$, the variance of an observation. Hence, $\sigma_\tau{}^2$ and $\sigma_\varepsilon{}^2$ are called"components of variance".This implies that observations y_{ij} 's are distributed with mean μ and common variance $\sigma_\tau{}^2 + \sigma_\varepsilon{}^2$.

Worked Example: A study was carried out to investigate the effect of lecture time in studying the "Calculus II " course on student's mark in the first exam in PPU. Thirty students were randomly selected from different classes of "Calculus II ". These students were randomly divided into three groups of 10 each. Three different times were randomly assigned to the three groups. The mark in the first exam was determined for each student and the data given as follows:

TABLE I. LECTURE TIME

Time I	*Time II*	*Time III*
11	33	25
25	21	25
43	15	35
44	32	29
17	16	32
50	32	24
22	27	42
41	18	50
29	24	32
33	32	28

where
Time I : the lecture time from 8: 00 to 9:15.
Time II :the lecture time from10:00 to 10:50.
Time III: the lecture time from 12:30 to 2:00.

The model for this experiment is
$y_{ij} = \mu + \tau_i + \varepsilon ij \ (i=1,...,3 \ ; \ j=1,...,10)$
where
y_{ij} is the j^{th} student's mark in the first exam in "calculus II" course, which was recorded on the i^{th} lecture time.
τ_i is the effect due to i^{th} lecture time.

Note : the data of TABLE I should be analyzed under Model I (fixed effects model) since the three lecture times are especially chosen by us to be of particular interest. In this example a=3, $n_1 = n_2 = n_3 = n = 10$; the results of the analysis of variance calculations are summarized in TABLE II.

TABLE II. ANOVA TABLE UNDER MODEL I

Source of Variation	*Sum of Squares*	*Df*	*Mean Square*	*F*	*Sig.*
Between Groups	315.267	2	157.633	1.653	.210
Within Groups	2574.100	27	95.337		
Total	2889.367	29			

In this example, the null hypothesis of interest is that all the lecture times have the same effect; that is, $H_0: \tau_1 = \tau_2 = \tau_3 = 0$ against the alternative H_1: at least one $\tau_i \neq 0$ *; (i=1, 2, 3).*
Suppose we choose the significance level of $\alpha = 0.05$, we find that the 95th percentage point of the F distribution with 2 and 27 degrees of freedom is 3.35. Since the value of the F statistic from Table 2.1.3 is $F = MS_\tau / MS_\varepsilon$ = 157.633/95.337 = 1.653 is less than 3.35 or $p -$ value = 0.21 (from the Table 2.1.3) is greater than $\alpha = 0.05$, we don't reject H_0 and we conclude that the lecture time doesn't affect the model.(i.e the effects of the three lecture times are the same) .
Randomized Complete Block Design: A RCBD is a restricted randomization design. In which the experimental units are first sorted into homogenous groups, called blocks, and the treatments are then assigned at random within the blocks.
In other words, The experimental units are matched according to a variable "we will call it later as a blocking variable "which the experimenter wishes to control. The experimental units are put into groups (blocks) of the same size as the number of treatment. The members of each block are then randomly assigned to different treatment groups.

Statistical Model and Analysis for RCBD: If we take n_i replications for each treatment or treatment combination in completely random manner, then the analysis of variance model for the experiment is given by

$$y_{ij} = \mu + \tau i + \beta_j + \varepsilon ij$$
$$(i=1,2,...,a \text{ and } j=1,2,...,b)$$

where:
y_{ij} is the observed value corresponding to the ith treatment and jth block.
μ is an overall mean ($-\infty < \mu < \infty$).
τ_i is the effect of the ith treatments.
β_j is the effect of the jth block.
εij is the usual normal independent $(0,\sigma_\varepsilon^2)$ random error term.

Notes:
- The model is the same as the two- way crossed classification, no interaction model with one observation per cell.
- There are three versions of model depending on whether blocks or treatments or both are chosen at random.
- The linear statistical model that we have used for the RCBD is completely additive.

In a Randomized Complete Block experiment, both blocks and treatments may be randomly chosen, and then we will have a model II or a random model. Here, all τi 's, β_j's, and εij 's are mutually and completely independent normal random variables with mean zero and variances $\sigma_\tau^2, \sigma_\beta^2$, and σ_ε^2, respectively.
The hypotheses on $\sigma_\beta^2$and σ_τ^2 can be tested by the same statistics as in the case when both blocks and treatments are fixed. And the blocks may be chosen randomly from a population of blocks, but the treatments may be fixed. In this case, we will have a model III or mixed model, where τ_i 's are fixed constants with the restriction that$\sum_{i=1}^{a} \tau_i = 0$; and the β_j 's and ε_{ij}'s are mutually and completely independent normal random variables with mean zero and variances σ_β^2, and σ_ε^2, respectively. The hypotheses about β_j's and τi 's can similarly be tested as in the case when both blocks and treatments are fixed. And if the treatments may be chosen at random from a population of treatments, but the blocks may be fixed. In this case, we again have a mixed model situation. The assumptions and tests of hypotheses are as given in the preceding case with the roles of the τi 's and β_j's being reversed.

Worked Example: An experiment was designed to study the effect of three different lecture times on the marks of students in the first exam in "calculus II" course. A Randomized Complete Block Design with three blocks was used and each lecture time was planted in each of the blocks.

Illustration: The experimental unit is the student, and 9 students are available for the study. The major of the student – usually – is highly correlated with the response variable (student's mark). Hence, it is desirable to block the 9 students into three groups of three students each, according to the major. Thus, the students majoring in applied mathematics will constitute block 1, the students majoring in computer science will constitute block 2, and the students majoring in engineering will constitute block 3. Within each block, the three lecture times are assigned at random to the three students, and the assignments from one block to another are made independently. The results are as follows:

TABLE III. LECTURE TIME EFFECT

Block(major)	***8:00-9:15***	***10:00-10:50***	***12:30-2:00***
1	50	33	25
2	43	32	32
3	44	27	24

The model for this experiment is

$$y_{ij} = \mu + \tau_i + \beta_j + \varepsilon ij (i = 1,2,3 \text{ and } j = 1,2,3)$$

Where
y_{ij} denotes the student's mark corresponding to the ith lecture time and jth block,
τi is the effect of the ith lecture time, and
β_j is the effect of the jth block(major).

Note: In this experiment, both blocks and treatments have a fixed effects fixed, so that the data from this experiment must be analyzed under Model I. The results of the analysis of variance calculations are summarized in Table 2.2.2

TABLE IV. ANOVA TESTS OF BETWEEN-SUBJECTS EFFECTS

Source	***Type III Sum of Squares***	***Df***	***Mean Square***	***F***	***Sig.***
Corrected Model	621.778a	4	155.444	11.856	.017
Intercept	10677.77	1	10677.77	814.40	.000
Block-major	34.889	2	17.444	1.331	.361
Treatment-time	586.889	2	293.444	22.381	.007
Error	52.444	4	13.111		
Total	11352.00	9			
Corrected Total	674.222	8			

In this example, the null hypothesis of interest is that all the lecture times have the same effect; that is, $H_0: \tau_1 = \tau_2 = \tau_3 = 0$ against the alternative H_1: at least one $\tau_i \neq 0$;(*i=1, 2, 3*)
Using $\alpha = 0.05$, the critical value of Fis $F_{2,4}^{0.95} = 6.94$. Because $F_\tau = MS_\tau / MS_\varepsilon = 22.381 > 6.94$ or the ratio of mean squares for treatments (Lecture Times) is highly significant (p = 0.007), we conclude that H_0 will be rejected, which mean that there is enough evidence of real treatment differences. The block (Major) effects seem to be insignificant (p = 0.361), if we tested the hypothesis (2.2.4), and there may be some question regarding the effectiveness of blocking.

IV. Latin Square Design(LSD)

we have just discussed and defined some vital concepts of experimental design, including randomization, replication, blocking, and other concepts, which lead to proper experimental design .Moreover, the thorough understanding of these concepts ensure the validity of the analysis of any experiment and increase its sensitivity. Next we have introduced some commonly experimental design, namely, the Completely Randomized Design (CRD) and the Randomized Complete Block Design (RCBD). Also we have discussed the description, advantages, disadvantages, the statistical model, the analysis of variance, and test of hypothesis of each of these two designs (CRD and RCBD).

In the Randomized Complete Block Design we have looked at blocking on a single factor. In other words, the RCBD was used to reduce experimental error by eliminating a source of variation in experimental units by utilizing the principle of blocking as we have just shown in section 2.2 .However there are times when there are two factors which we want to block simultaneously, that is, the experimental units are grouped in blocks in two different ways .Therefore, two different sources of variation can be eliminated by using two-way or double blocking on the experimental units. The design will accomplish this is called the Latin Square Design.

In this chapter we will discuss – in details – the design and the analysis of Latin Square experiments , we shall introduce the description, the statistical model, the analysis of variance, and testing hypothesis of this design.

Description of Latin Square Design (LSD): A Latin Square for p treatments, or a p×p Latin Square, is a square matrix with p rows and p columns. Each of the resulting p^2 cells contains one of the p letters. Each letter corresponds to one of the treatments and each letter occurs once and only once in each row and in each column.

Clearly, a Latin Square Design has the following features: Treatments are arranged in rows and columns. The number of rows, columns and treatments must all be the same. There are two blocking variables, each containing p classes. Each row contains every treatment, and also each column. In other words each row and each column in this design contains all treatments, that is, each class of each blocking variable constitute a replication.

Remark: Latin Squares are often used to study the effects of three factors, where the factors corresponding to the rows and columns are of interest in themselves and not introduced for the main purpose of reducing experimental error. We note that in a Latin Square there are only p^2 experimental units to be used in the experiment instead of the p^3 possible experimental units needed in a complete three-way layout. Thus, the use of the Latin Square Design results in the savings in observations by a factor $1/p$ observations over the complete three layout.

There are some advantages of Latin Square Design, one of them is that –with the Latin Square Design- we can control variation in two directions. In other words, the use of two blocking variables often permits greater reductions in the variability of experimental errors than can be obtained with either blocking variable alone. However, in this design the number of classes of each blocking variable must equal the number of treatments and this restriction is often difficult to meet in practice. Also, among the disadvantages of LSD, the use of this design will lead to a very small number of degrees of freedom for experimental error when only a few treatments are studied. On the other hand, when many treatments are studied, the degrees of freedom for experimental error may be larger than necessary. The assumptions of LSD model are restrictive. For example, there are no interaction between either blocking variable and treatments, also none between the two blocking variables (Rows and Columns).

Latin Square Model: A Latin square design model involves the main effect of row blocking variable, denoted by α_i, the main effect of the column blocking variable, denoted by β_j, and the treatment main effect, denoted by τ_k. It is assumed that no interactions exist between these three variables. Thus, the model employed is an additive one. For the case of fixed treatment and block effect, the model is:

$$y_{ijk} = \mu+\alpha_i+\beta_j+\tau_k+\varepsilon_{ijk} \begin{cases} i = 1,2, \dots, p \\ j = 1,2, \dots, p \\ k = 1,2, \dots, p \end{cases}$$

y_{ijk} denotes the observed value corresponding to the i-throw, the jth column, and the kth treatment; $-\infty < \mu < \infty$ is the overall mean.

α_i is the effect of the i-th row, β_j is the effect of the j-th column, and τ_k is the effect of the k-th treatment. All of these effects (α_i, β_j , τ_k) are constants subject to the restrictions$\sum_{i=1}^{p} \alpha_i = 0$.

ε_{ijk} is the random error. These are independent $N(0,\sigma_\varepsilon^2)$.

We note again as we have mentioned before that the number of classes for each of the two blocking variables is the same as the number of treatments, so that the total number of experimental trials is p^2 .

Note that if the treatment effects are random, the only change in model (3.3.1) is that the τ_k now are independent $N(0,\sigma_\tau^2)$ and are independent of the ε_{ijk}. There is also, only one observation in each cell, so that only two of the three subscripts i , j and k are needed to denote a particular observation.

$$\bar{y}_{...} = \left(\sum_{i=1}^{p}\sum_{j=1}^{p} y_{ijk}\right)/p^2 = y_{...}/p$$

The corresponding degrees of freedom are partitioned as

p^2-1 = (p – 1) + (p - 1) + (p - 1) + (p - 1)(p - 2)

Note: The identity is valid since all the cross – product terms are equal to zero. The identity states that we have partitioned the total variation SST into the following components:

SS_R : denotes the sum of squares due to the α_i 's, representing the variation in the y_{ijk} due to the row effect.

SS_R : denotes the sum of squares due to the β_j 's, representing the variation in the y_{ijk} due to the column effect. SS_τ : denotes the sum of squares due to the τ_k 's, representing the variation in the y_{ijk} due to the treatment effect. Analysis of Variance and Tests:

TABLE V. ANOVA TABLE FOR LATIN SQUARE DESIGN WITH FIXED EFFECTS

Source of Variation	*Degrees of Freedom*	*Sum of Squares*	*Mean Square*	*Modle I*
Treatment	p-1	SS_τ	MS_τ	$\sigma_\varepsilon^2+\frac{P}{P-1}\sum\tau_k 2$
Row	p-1	SS_R	MS_R	$\sigma_\varepsilon^2+\frac{P}{P-1}\sum\alpha_i 2$
Column	p-1	SS_C	MS_C	$\sigma_\varepsilon^2+\frac{P}{P-1}\sum\beta_j$
Error	(p-1)(p-2)	SS_E	MS_E	σ_ε^2
Total	p^2-1	SS_T		

In this table the mean squares are

$MS_\tau = SS_\tau/ (p-1) = p (\sum_{k=1}^{p} \bar{y}..k-\bar{y}...) 2 / (p -1)$

$MS_R = SS_R /(p-1) = p \sum_{i=1}^{p}(\bar{y}i.. - \bar{y}...) 2/ (p - 1)$

$MS_C = SS_C/(p-1) = p (\sum_{j=1}^{p}(\bar{y}.j. - \bar{y}...)^2 /(p - 1)$

$$MS_E = SS_E/ (p-1)(p-2) = \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}(y_{ijk} - yi..-y.j.-y..k +2y...)^2 / (p-1)(p-2)$$

Test for treatment effects :To test for treatment effects in Latin square model (3.3.1) with fixed effects H_0: $\tau_1 = \tau_2 = ... = \tau_P = 0$, versus H_1: $\tau_K \neq 0$ for at least one k, $k = 1, 2, ... ,p$ we see that the appropriate test statistic is: $F^* = MS_\tau / MS_E$ The appropriate decision rule to control the risk of a type 1 of error at α is:

If $F^* \leq F[1-\alpha; p-1, (p-1)(p-2)]$, conclude H_0

If $F^* > F[1-\alpha; p-1, (p-1)(p-2)]$, conclude H_1

Worked Example: An oil company tested four different blends of gasoline for fuel efficiency according to a Latin square design in order to control for the variability of four different drivers and four different models of cars. Fuel efficiency was measured in miles per gallon (mpg) after driving cars over a standard course.

TABLE VI. CAR MODEL VS DRIVER

	Car Model			
Driver	***I***	***II***	***III***	***IV***
1	D 15.5	B 33.9	C 13.2	A 29.1
2	B 16.3	C 26.6	A 19.4	D 22.8
3	C 10.8	A 31.1	D 17.1	B 30.3
4	A 14.7	D 34.0	B 19.7	C 21.6

TABLE VI. ANOVA TABLE FOR CAR MODEL VS DRIVERS

Source of Variation	*Degrees of Freedom*	*Sums of Squares*	*Mean Square*	*F value*	p value
Blend	3	108.982		9.15	0.012
Car Model	3	736.912	245.637	61.90	0.000
Driver	3	5.897	36.327	0.50	0.699
Error	6	23.809	3.968		
Total	15	875.599			

We conclude that the blends are significantly different at the 5% level, but not at the 1% level. It is not surprising that the fuel efficiencies vary among various models of cars (which might range from small compacts to large RVs). It does not appear, however, that there are any significant differences among our Drivers. (In the general population some drivers are easier on fuel than others; perhaps the drivers for this study have been carefully trained so that their driving styles are similar.)

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